Lecture 3 — Examples of Model Structures; Quillen Functors and Equivalences

Examples: Top and sSet

Definition 1. Let **Top** be the category of topological spaces.

1. **(Weak homotopy equivalences.)** A continuous map $f\colon X \to Y$ is a **weak homotopy equivalence** if it induces isomorphisms on the set of connected components $\pi_0 f : \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ and on all homotopy groups,
$$\pi_n f\colon \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x)),$$
for all $n \geq 1$ and all choices of basepoint $x \in X$.

2. **(Generating trivial cofibrations; Serre fibrations.)** Call
$$J \ := \ \left\{ j_n\colon D^n \times \{0\} \hookrightarrow D^n \times I \right\}_{n\in\mathbb{N}}$$
the set of **generating trivial cofibrations**. A **Serre fibration** is a map $p\colon A \to B$ in RLP(J), i.e. every solid-arrow square
$$\begin{array}{ccc} D^n & \longrightarrow & A \\ {\scriptstyle j_n}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ D^n \times I & \longrightarrow & B \end{array}$$
admits a lift.

3. **(Generating cofibrations; relative cell complexes.)** Call
$$I \ := \ \left\{ i_m\colon S^{m-1} \hookrightarrow D^m \right\}_{m\in\mathbb{N}}$$
the set of **generating cofibrations**, where $S^{-1} = \varnothing$. An m**-cell attachment** to X is the pushout of some $i_m \in I$ along a map $\varphi\colon S^{m-1} \to X$:
$$\begin{array}{ccc} S^{m-1} & \xrightarrow{\ \varphi\ } & X \\ {\scriptstyle i_m}\big\downarrow & & \big\downarrow \\ D^m & \longrightarrow & X \sqcup_{S^{m-1}} D^m. \end{array}$$
A map $f\colon X \to Y$ is a **relative cell complex** if it is given by a sequence (possibly infinite) of m-cell attachments of the form
$$\begin{array}{ccc} \sqcup_j S^{m_j-1} & \longrightarrow & X_{k-1} \\ {\scriptstyle \sqcup_j i_{m_j}}\big\downarrow & & \big\downarrow \\ \sqcup_j D^{m_j} & \longrightarrow & X_k, \end{array}$$

as

$$X = X_0 \to X_1 \to X_2 \to \cdots \to Y = \varinjlim_k X_k.$$

Definition 2. **Top** admits a model structure $\mathbf{Top}_Q$. A morphism in $\mathbf{Top}_Q$ is:

1. a **weak equivalence** if it is a weak homotopy equivalence;
2. a **fibration** if it is a Serre fibration;
3. a **cofibration** if it is a retract of a relative cell complex.

Definition. $\mathbf{Top}_*$ admits the model structure $\mathbf{Top}_{*Q}$. If $U\colon \mathbf{Top}_* \to \mathbf{Top}$ is the forgetful functor, $(X,*) \mapsto X$, then $f \in W(\mathbf{Top}_{*Q})$ (respectively in $\mathrm{Fib}(\mathbf{Top}_{*Q}), \mathrm{Cof}(\mathbf{Top}_{*Q})$) if and only if $Uf \in W(\mathbf{Top}_Q)$ (respectively in $\mathrm{Fib}(\mathbf{Top}_Q), \mathrm{Cof}(\mathbf{Top}_Q)$).

Definition 3. The **simplex category** Δ has objects $[m]$, $m \in \mathbb{N}$, where we think of $[m]$ as the totally ordered set $\{0 \to 1 \to \cdots \to m\}$; morphisms are non-decreasing maps. A **simplicial set** S is a functor $S\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$. Simplicial sets form the category **sSet**. There is a functor $\Delta \to \mathbf{sSet}$ sending

$$[m] \;\longmapsto\; \Delta[m] \;:=\; \mathrm{Hom}_\Delta(-,[m]) \;:\; \Delta^{\mathrm{op}} \to \mathbf{Set}.$$

Definition 4. The **topological n-simplex** is

$$\Delta^n \;:=\; \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \textstyle\sum_i x_i = 1,\; x_i \geq 0 \;\forall\, i \right\}.$$

In low degrees: $\Delta^0 = \{*\}$, $\Delta^1 = I$, Δ^2 is the topological triangle, and Δ^3 is the topological tetrahedron.

Definition. For $X \in \mathbf{Top}$, a **singular m-simplex** of X is a map $\sigma\colon \Delta^m \to X$. Define the set of singular m-simplices of X as

$$\mathrm{Sing}_m X \;:=\; \mathrm{Hom}_{\mathbf{Top}}(\Delta^m, X).$$

The assignment $[m] \mapsto \mathrm{Hom}_{\mathbf{Top}}(\Delta^m, X)$ is contravariant in $[m]$ via precomposition, and defines a functor $\mathrm{Sing}\, X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$, i.e. a simplicial set. This is the **singular simplicial complex** of X. The construction is natural in X, giving a functor $\mathrm{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$.

Definition. $\mathrm{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$ admits a **left adjoint** $|\text{-}|\colon \mathbf{sSet} \to \mathbf{Top}$, $S \mapsto |S|$, called **geometric realization**. In particular $|\Delta[m]| = \Delta^m$.

Definition 5. For $0 \leq i \leq m$, the **(m,i)-horn** is the simplicial subset

$$\Lambda^i[m] \;\hookrightarrow\; \Delta[m],$$

where, recalling that $\Delta[m]\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ sends $[k]$ to $\mathrm{Hom}_\Delta([k],[m])$, the subset $\Lambda^i[m]$ consists of those non-decreasing maps $\alpha : [k] \to [m]$ such that $\alpha\left([k]\right) \cup \{i\} \neq [m]$.

Definition 6. A map $\varphi\colon S \to T$ in **sSet** is a **Kan fibration** if it has the right lifting property with respect to all horns, i.e. every solid-arrow square

$$\begin{array}{ccc} \Lambda^i[m] & \longrightarrow & S \\ \big\downarrow & \nearrow & \big\downarrow \varphi \\ \Delta[m] & \longrightarrow & T \end{array}$$

admits a diagonal filler.

Definition 7. The category **sSet** admits a model structure $\mathbf{sSet}_Q$. A morphism $\varphi\colon S \to T$ in $\mathbf{sSet}_Q$ is:

- a **weak equivalence** ($\varphi \in W$) if $|\varphi|\colon |S| \to |T|$ is a weak homotopy equivalence in $\mathbf{Top}_Q$;
- a **fibration** if φ is a Kan fibration;
- a **cofibration** if φ is a monomorphism, equivalently a degreewise inclusion.

Quillen functors and adjunctions

Definition 8. Let C and D be model categories.

1. A functor $F\colon C \to D$ is a **left Quillen functor** if:
   - F is a left adjoint, and
   - F preserves cofibrations and trivial cofibrations.
2. A functor $U\colon D \to C$ is a **right Quillen functor** if:
   - U is a right adjoint, and
   - U preserves fibrations and trivial fibrations.
3. Let $F \dashv U$, with $F\colon C \to D$ and $U\colon D \to C$, and let $\varphi\colon D(FA, B) \xrightarrow{\sim} C(A, UB)$ be the adjunction isomorphism. The triple (F, U, φ) is a **Quillen adjunction** if F is a left Quillen functor.

Lemma 9. *(F, U, φ) is a Quillen adjunction if and only if U is a right Quillen functor.*

Total derived functors

Definition 10. Let C, D be model categories.

1. If $F\colon C \to D$ is a left Quillen functor,

   $$\mathbf{L}F \;:\; \operatorname{Ho} C \xrightarrow{\operatorname{Ho} Q} \operatorname{Ho} C_c \xrightarrow{\operatorname{Ho} F} \operatorname{Ho} D$$

   is the **total left derived functor** of F. For a natural transformation $\tau\colon F \to F'$ of left Quillen functors, the **total left derived natural transformation** is $(\mathbf{L}\tau)_X \coloneqq \tau_{QX}$.

2. If $U\colon D \to C$ is a right Quillen functor,

$$\mathbf{R}U \;:\; \operatorname{Ho} D \xrightarrow{\operatorname{Ho} R} \operatorname{Ho} D_f \xrightarrow{\operatorname{Ho} U} \operatorname{Ho} C$$

is the **total right derived functor** of U.

Remark. *To define* $\mathbf{L}F$ *it suffices that* F *send weak equivalences between cofibrant objects to weak equivalences in* D*. Vice versa for* $\mathbf{R}U$*.*

Lemma 11. *Let* $(F, U, \varphi)\colon C \rightleftarrows D$ *be a Quillen adjunction. Then there is a* ***derived adjunction***

$$\operatorname{Ho} C \underset{\mathbf{R}U}{\overset{\mathbf{L}F}{\rightleftarrows}} \operatorname{Ho} D,$$

with $\mathbf{R}\varphi\colon \operatorname{Ho} D(FQX, Y) \xrightarrow{\sim} \operatorname{Ho} C(X, URY)$*, and we write* $L(F, U, \varphi) = (\mathbf{L}F, \mathbf{R}U, \mathbf{R}\varphi)$*.*

Definition 12. (F, U, φ) is a **Quillen equivalence** if for every cofibrant $X \in C_c$ and every fibrant $Y \in D_f$,

$$f\colon FX \to Y \;\in W_D \iff \varphi(f)\colon X \to UY \;\in W_C.$$

Proposition 13. (F, U, φ) *is a Quillen equivalence if and only if* $L(F, U, \varphi)$ *is an adjoint equivalence* $\operatorname{Ho} C \simeq \operatorname{Ho} D$*.*

Definition 14. A functor F **preserves** a property P for morphisms if $f \in P \Rightarrow Ff \in P$. A functor F **reflects** P if $Ff \in P \Rightarrow f \in P$.

Corollary 15. *The following are equivalent for a Quillen adjunction* (F, U, φ)*:*

1. (F, U, φ) *is a Quillen equivalence.*

2. F *reflects weak equivalences between cofibrant objects, and for every fibrant* $Y \in D_f$ *the composite*

$$FQUY \xrightarrow{F(q_{UY})} FUY \xrightarrow{\varepsilon_Y} Y$$

   is a weak equivalence.

3. U *reflects weak equivalences between fibrant objects, and for every cofibrant* $X \in C_c$ *the composite*

$$X \xrightarrow{\eta_X} UFX \xrightarrow{U(r_{FX})} URFX$$

   is a weak equivalence.

Theorem 16. *The adjunction* $|\text{-}| \dashv \operatorname{Sing}$*,*

$$\mathbf{sSet}_Q \underset{\operatorname{Sing}}{\overset{|\text{-}|}{\rightleftarrows}} \mathbf{Top}_Q,$$

is a Quillen equivalence. In particular, $\operatorname{Ho}(\mathbf{sSet}_Q) \simeq \operatorname{Ho}(\mathbf{Top}_Q)$*.*